

REPRESENTATIONS OF TWISTED TENSOR PRODUCTS

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ABSTRACT. We obtain a faithful representation of the twisted tensor product $B \otimes_\chi A$ of unital associative algebras, when B is finite dimensional. This generalizes the representations of [3] where $B = K[X]/\langle X^2 - X \rangle$, [5] where $B = K[X]/\langle X^n \rangle$ and [7] where $B = K^n$. Furthermore, we establish conditions to extend twisted tensor products $B \otimes_\chi A$ and $C \otimes_\psi A$ to a twisted tensor product $(B \times C) \otimes_\varphi A$.

Introduction

Let k be a commutative ring and let A, B be unitary k -algebras. A *twisted tensor product of A with B* (over k) is an algebra structure defined on $A \otimes_k B$, such that the canonical maps $i_A: A \rightarrow A \otimes_k B$ and $i_B: B \rightarrow A \otimes_k B$ are algebra maps satisfying $a \otimes b = i_A(a)i_B(b)$. This structure has been studied by many people with different motivations (see for instance [2], [1], [4], [6], [9], [10], [11]). On one hand it is the most general solution to the problem of factorization of structures in the setting of associative algebras. Consequently, a number of examples of classical and recently defined constructions in ring theory fits into this construction. For instance, Ore extensions, skew group algebras, smash products, etcetera. On the other hand it has been proposed as the natural representative for the cartesian product of noncommutative spaces, this being based on the existing duality between the categories of algebraic affine spaces and commutative algebras, under which the cartesian product of spaces corresponds to the tensor product of algebras. Besides that, twisted tensor products arise as a useful tool for building algebras starting with simpler ones.

Given algebras A and B , a basic problem is to determine families of twisted tensor products of A with B (ideally all of them) and classify them up to a natural equivalence relation. In general this is a very difficult problem. For instance, in [5, Corollary 4.2] it was proven that determining all the twisted tensor products of A with $k[X]/\langle X^5 \rangle$ with the form

$$(1 \otimes X)(a \otimes 1) = a \otimes X + \gamma_1^2(a) \otimes X^2 + \gamma_1^3(a) \otimes X^3 + \gamma_1^4(a) \otimes X^4,$$

where $\gamma_i^j: A \rightarrow A$ ($j = 2, 3, 4$) are k -linear maps, is equivalent to find all the pairs (δ_1, δ_2) of derivations of A , such that $[\delta_1] \cup [\delta_2] = 0$, where $[\delta_i]$ is the class of δ_i in the cohomology group $H^1(A)$ and \cup denotes the cup product. In spite of these difficulties, this problem was considered for different types of algebras in [3], [5], [7] and [8]. When k is a field and the algebras involved are finite dimensional over k , as in the these papers, given twisted tensor product C of A with B and fixed a basis $\{b_1, \dots, b_n\}$, there exists maps $\gamma_i^j: A \rightarrow A$, satisfying suitable condition, such that

$$\chi(1 \otimes a)(b_i \otimes 1) = \sum_{j=1}^n b_j \otimes \gamma_i^j(a), \quad \text{for all } a \in A \text{ and all } i.$$

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In Corollary 2.1 and Proposition 2.5 we show that these conditions are satisfied if and only if the map $\hat{\rho}_\chi$ and $\hat{\phi}_\chi$, introduced in the statements of these results, are matricial representations of algebras. The first step in the study of the twisted tensor products considered in [3], [5], [7] and [8] was determine the conditions required to the maps γ_i^j by means of direct computations. Using the fact that $\hat{\rho}_\chi$ and $\hat{\phi}_\chi$ must be representations, these conditions arise naturally in each of the examples, as is shown in the last section.

1 Preliminaries

Let K be a commutative ring with 1. All the maps considered in this notes are K -linear maps, the symbol \otimes denotes the tensor product over K , by an algebra we understand an associative unital K -algebra and the algebra morphisms are unital.

1.1 Twisting maps

Let A and B be algebras, and let μ_A and μ_B the multiplication maps of A and B , respectively. A *twisted tensor product* of B with A is an algebra structure over the K -module $B \otimes A$, such that the canonical maps

$$i_B: B \longrightarrow B \otimes A \quad \text{and} \quad i_A: A \longrightarrow B \otimes A$$

are morphisms of algebras, and $i_B(b)i_A(a) = b \otimes a$ for all $b \in B$ and $a \in A$.

Given a twisted tensor product of B with A , the map

$$\chi: A \otimes B \longrightarrow B \otimes A,$$

defined by $\chi := \mu \circ (i_A \otimes i_B)$, satisfies:

- (1) $\chi(1 \otimes b) = b \otimes 1$ and $\chi(a \otimes 1) = 1 \otimes a$, for all $a \in A$ and $b \in B$,
- (2) $\chi \circ (\mu_A \otimes B) = (B \otimes \mu_A) \circ (\chi \otimes A) \circ (A \otimes \chi)$,
- (3) $\chi \circ (A \otimes \mu_B) = (\mu_B \otimes A) \circ (B \otimes \chi) \circ (\chi \otimes B)$.

A map fulfilling these conditions is called a *twisting map*. Conversely, if

$$\chi: A \otimes B \longrightarrow B \otimes A$$

is a twisting map, then $A \otimes B$ becomes a twisted tensor product via

$$\mu_\chi := (\mu_B \otimes \mu_A) \circ (B \otimes \chi \otimes A).$$

This algebra will be denote $B \otimes_\chi A$. It is evident that these constructions are inverse one of each other.

Remark 1.1. The right action of A on $B \otimes_\chi A$ induced by the canonical map of A into $B \otimes_\chi A$, is the canonical right action of A on $B \otimes_\chi A$. Similarly, the left action of B on $B \otimes_\chi A$ induced by the canonical map of B in $B \otimes_\chi A$ is the canonical left action of B on $B \otimes_\chi A$.

Remark 1.2. It is easy to check that the twisted tensor product $B \otimes_\chi A$ has the following universal property: given morphisms of algebras $\varphi: A \rightarrow C$ and $\psi: B \rightarrow C$, there exists a unique morphisms of algebra $\Psi: B \otimes_\chi A \rightarrow C$ such that $\Psi \circ i_A = \varphi$ and $\Psi \circ i_B = \psi$ if and only if

$$\mu_C \circ (\varphi \otimes \psi) = \mu_C \circ (\psi \otimes \varphi) \circ \chi.$$

Consequently, if C is a twisted tensor product $E \otimes_\varpi D$, then given morphisms of algebras $f: A \rightarrow D$ and $g: B \rightarrow E$, the map

$$g \otimes f: B \otimes_\chi A \longrightarrow E \otimes_\varpi D$$

is a morphism of algebras if and only if $\varpi \circ (f \otimes g) = (g \otimes f) \circ \chi$.

1.2 Twisting maps of finite dimensional algebras

From now on we assume that K is a field and that B is a finite dimensional algebra over K . Moreover we fix a basis $\mathcal{B} := \{b_1, \dots, b_n\}$ of B . Recall that the structure constants λ_{ij}^k ($1 \leq i, j, k \leq n$) of B with respect to \mathcal{B} are the scalars determined by the equalities

$$b_i b_j = \sum_{k=1}^n \lambda_{ij}^k b_k.$$

Since the multiplication B is associative, the λ_{ij}^k 's satisfy

$$\sum_{l=1}^n \lambda_{ij}^l \lambda_{lk}^m = \sum_{l=1}^n \lambda_{jk}^l \lambda_{il}^m.$$

Moreover, since B is unital, if $1 = \sum_j \alpha_j b_j$, then

$$\sum_{j=1}^n \alpha_j \lambda_{ji}^k = \sum_{j=1}^n \alpha_j \lambda_{ij}^k = \delta_{ki}.$$

A map $\chi: A \otimes B \longrightarrow B \otimes A$ determine unique maps $\gamma_i^j: A \longrightarrow A$ ($0 \leq i, j < n$), such that

$$\chi(a \otimes b_i) = \sum_{j=1}^n b_j \otimes \gamma_i^j(a), \quad \text{for all } i. \quad (1.1)$$

Proposition 1.3. *The map χ is a twisting map if and only if the functions γ_j^i satisfy the following properties:*

- (1) $\gamma_i^j(1) = \delta_{ij} 1$,
- (2) $\gamma_i^k(aa') = \sum_j \gamma_j^k(a) \gamma_i^j(a')$,
- (3) $\alpha_k \text{id} = \sum_i \alpha_i \gamma_i^k$,
- (4) $\sum_k \lambda_{ij}^k \gamma_k^m = \sum_k \sum_l \lambda_{kl}^m \gamma_j^l \circ \gamma_i^k$.

More precisely, conditions (1) and (2) are satisfied if and only if χ is compatible with the algebra structure of A , and conditions (3) and (4) are satisfied if and only if it is compatible with that of B .

Proof. Left to the reader. □

2 The canonical representation

Recall that the regular left representation of a K -algebra C is the morphism $l: C \longrightarrow \text{End}_K(C)_C$ defined by $l(c) := l_c$, where $l_c \in \text{End}_K(C)$ is the map given by $l_c(d) := cd$ for all $d \in C$. Since $l_c(1) = c$ for all $c \in C$, this is a faithful representation. Consequently, for each algebra L the map $l^L: B^{\text{op}} \otimes L \longrightarrow \text{End}(B^{\text{op}} \otimes L)_L$, given by

$$\begin{array}{ccc} B^{\text{op}} \otimes L & \xrightarrow{l^L} & \text{End}_K(B^{\text{op}} \otimes L)_L \\ & \searrow l \quad \nearrow i & \\ & \text{End}_K(B^{\text{op}} \otimes L)_{B^{\text{op}} \otimes L} & \end{array}$$

in which i is the canonical inclusion, is a faithful representation, called the *canonical representation* of $B^{\text{op}} \otimes E$.

From now on we set $E := \text{End}_K(A)$. In the rest of the section we assume that we are in the conditions of Subsection 1.2 and we denote by \mathcal{B}' the basis $\{b_1^{\text{op}}, \dots, b_n^{\text{op}}\}$ of B^{op} .

Proposition 2.1. *Items (3) and (4) of Proposition 1.3 are satisfied if and only if*

$$\begin{array}{ccc} B^{\text{op}} & \xrightarrow{\rho_\chi} & B^{\text{op}} \otimes E \\ b_k^{\text{op}} & \longmapsto & \sum_j b_j^{\text{op}} \otimes \gamma_k^j \end{array}$$

is a morphism of K -algebras.

Proof. This follows by a direct computation. \square

Corollary 2.2. *Consider the right E -linear bijection*

$$\begin{array}{ccc} B^{\text{op}} \otimes E & \xrightarrow{\xi_{\mathcal{B}'}} & E^n, \\ b_k^{\text{op}} \otimes 1 & \longmapsto & e_k \end{array}$$

where $\{e_1, \dots, e_n\}$ is the canonical basis of E^n . Items (3) and (4) of Proposition 1.3 are satisfied if and only if the map

$$\begin{array}{ccc} B^{\text{op}} & \xrightarrow{\ddot{\rho}_\chi} & \text{End}(E^n)_E \\ b^{\text{op}} & \longmapsto & \xi_{\mathcal{B}'} \circ l^E(\rho_\chi(b^{\text{op}})) \circ \xi_{\mathcal{B}'}^{-1} \end{array}$$

is a representation of B^{op} or, equivalently, if the map $\hat{\rho}_\chi: B^{\text{op}} \rightarrow M_n(E)$, that sends $b^{\text{op}} \in B^{\text{op}}$ to the matrix of $\ddot{\rho}_\chi(b^{\text{op}})$ with respect to the canonical basis, is a matrix representation.

Proof. Since l^E is faithful and $\xi_{\mathcal{B}'}$ is a bijective map, this is an immediate consequence of Proposition 2.1. \square

Definition 2.3. By definition, for each $b_k \in \mathcal{B}$ the structure matrix of b_k with respect to (B, \mathcal{B}) is the matrix $[b_k]_{\mathcal{B}} := (\lambda_{kj}^i)_{ij}$, where the λ_{uv}^w 's ($1 \leq u, v, w \leq n$) are the structure constants of B with respect to \mathcal{B} .

Remark 2.4. A direct computation shows that

$$\hat{\rho}_\chi(b_k^{\text{op}}) = \begin{pmatrix} \sum_{l=1}^n \lambda_{1l}^1 \gamma_k^l & \dots & \sum_{l=1}^n \lambda_{nl}^1 \gamma_k^l \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^n \lambda_{1l}^n \gamma_k^l & \dots & \sum_{l=1}^n \lambda_{nl}^n \gamma_k^l \end{pmatrix} = \sum_{j=1}^n \gamma_k^j [b_j^{\text{op}}]_{\mathcal{B}} \quad \text{for all } k,$$

where $[b_j^{\text{op}}]_{\mathcal{B}}$ is the structure matrix of b_j^{op} with respect to $(B^{\text{op}}, \mathcal{B})$.

Proposition 2.5. *Conditions (1) and (2) of Proposition 1.3 are fulfilled if and only if the function $\phi_\chi: A \rightarrow \text{End}(B \otimes A)_A$, defined by $\phi_\chi(a)(b_k \otimes 1) := \sum_j b_j \otimes \gamma_k^j(a)$, is a morphism of algebras or, equivalently, if the map $\hat{\phi}_\chi: A \rightarrow M_n(A)$, that sends $a \in A$ to the matrix of $\phi_\chi(a)$ with respect to the basis $\mathcal{B} \otimes 1 := \{b_1 \otimes 1, \dots, b_n \otimes 1\}$ of $B \otimes A$, is a matrix representation.*

Proof. This follows by a direct computation. \square

Remark 2.6. From equality (1.1) it follows that $\phi_\chi(a)(b \otimes 1) = \chi(a \otimes b)$ for all $b \in B$. In particular $\phi_\chi(a)(1 \otimes 1) = 1 \otimes a$, which implies that ϕ_χ is injective.

Corollary 2.7. *Consider the isomorphisms of right A -modules*

$$\begin{array}{ccc} B \otimes A & \xrightarrow{\xi_B} & A^n, \\ b_k \otimes 1 & \longmapsto & f_k \end{array}$$

where $\{f_1, \dots, f_n\}$ is the canonical basis of A^n . If χ is a twisting map, then the function

$$\begin{array}{ccc} B \otimes_\chi A & \xrightarrow{\chi l_B^A} & \text{End}(A^n)_A \\ b \otimes a & \longmapsto & \xi_B \circ l_{b \otimes 1} \circ \phi_\chi(a) \circ \xi_B^{-1} \end{array}$$

is a representation of $B \otimes_\chi A$ in $\text{End}(A^n)_A$.

Proof. This follows from Proposition 2.5, the fact that under condition (4) of Proposition 1.3

$$\phi_\chi(a) \circ l_{b_i \otimes 1} = \sum_j l_{b_j \otimes 1} \circ \phi_\chi(\gamma_i^j(a)),$$

and the fact that $\xi_{B'}$ is a bijective map. \square

Remark 2.8. A direct computation shows that

$$l_{b \otimes 1} \circ \phi_\chi(a)(b' \otimes 1) = (b \otimes 1)\chi(a \otimes b') \quad \text{for all } a \in A \text{ and } b, b' \in B.$$

Consequently $l_{b \otimes 1} \circ \phi_\chi(a)(1 \otimes 1) = b \otimes a$, which implies that χl_B^A is an injective map.

Corollary 2.9. *If χ is a twisting map, then the formulas*

$$\varphi_\chi(a) := \begin{pmatrix} \gamma_1^1(a) & \dots & \gamma_n^1(a) \\ \vdots & \ddots & \vdots \\ \gamma_1^n(a) & \dots & \gamma_n^n(a) \end{pmatrix} \quad \text{and} \quad \varphi_\chi(b_k) := \begin{pmatrix} \lambda_{k1}^1 \cdot 1_A & \dots & \lambda_{kn}^1 \cdot 1_A \\ \vdots & \ddots & \vdots \\ \lambda_{k1}^n \cdot 1_A & \dots & \lambda_{kn}^n \cdot 1_A \end{pmatrix},$$

for all $a \in A$ and $1 \leq k \leq n$, define a faithful representation $\varphi_\chi: B \otimes_\chi A \longrightarrow M_n(A)$.

Proof. It suffices to note that the matrices of $\chi l_B^A(1 \otimes a)$ and $\chi l_B^A(b_k \otimes 1)$ in the canonical basis of A^n , are the matrices $\varphi_\chi(a)$ and $\varphi_\chi(b_k)$, respectively. \square

3 Extensions of twisting maps

Let A , B and C be K -algebras with B and C finite dimensional. Write $D := B \times C$. Let $i_B: B \rightarrow D$, $i_C: C \rightarrow D$, $p_B: D \rightarrow B$ and $p_C: D \rightarrow C$ be the canonical maps. In this section we study on one hand twisting maps $\psi: A \otimes D \longrightarrow D \otimes A$ such that the map $\Theta := (p_B \otimes A) \circ \psi \circ (A \otimes i_B)$ is a twisting map, and on the other hand twisting maps such that both maps, Θ and $\Upsilon := (p_C \otimes A) \circ \psi \circ (A \otimes i_C)$, are twisting maps.

Fix basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_m\}$ of B and C , respectively, and let \mathcal{D} denote the ordered basis $\{b_1, b_2, \dots, b_n, c_1, \dots, c_m\}$ of \mathcal{D} , where we identify b_i with $(b_i, 0)$ and c_i with $(0, c_i)$. So, $\mathcal{D} = \{d_1, \dots, d_{m+n}\}$, where

$$d_i := \begin{cases} b_i & \text{if } i \leq n, \\ c_{i-n} & \text{if } i > n. \end{cases}$$

The structure matrices of the elements of D with respect to (D, \mathcal{D}) are

$$[b_k]_{\mathcal{D}} = \begin{pmatrix} [b_k]_{\mathcal{B}} & 0 \\ 0 & 0 \end{pmatrix} \quad (1 \leq k \leq n) \quad \text{and} \quad [c_l]_{\mathcal{D}} = \begin{pmatrix} 0 & 0 \\ 0 & [c_l]_{\mathcal{C}} \end{pmatrix} \quad (1 \leq l \leq m).$$

Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ be the scalars such that

$$1_D = (1_B, 1_C) = \sum_{k=1}^n \alpha_k b_k + \sum_{l=1}^m \beta_l c_l.$$

Each map $\psi: A \otimes D \longrightarrow D \otimes A$ determines uniquely functions $\tilde{\gamma}_i^j$ ($1 \leq i, j < m+n$), such that

$$\psi(a \otimes d_i) = \sum_{j=1}^{m+n} d_j \otimes \tilde{\gamma}_i^j(a) \quad (3.2)$$

Let λ_{ij}^k ($1 \leq i, j, k \leq n$) and η_{ij}^k ($1 \leq i, j, k \leq m$) be the structure constants of B with respect to \mathcal{B} and of C with respect to \mathcal{C} , respectively. Write

$$B_k^{(1)} := \begin{pmatrix} \sum_{l=1}^n \lambda_{1l}^1 \tilde{\gamma}_k^l & \cdots & \sum_{l=1}^n \lambda_{nl}^1 \tilde{\gamma}_k^l \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^n \lambda_{1l}^n \tilde{\gamma}_k^l & \cdots & \sum_{l=1}^n \lambda_{nl}^n \tilde{\gamma}_k^l \end{pmatrix} \in M_n(E) \quad (1 \leq k \leq n),$$

$$B_k^{(2)} := \begin{pmatrix} \sum_{l=1}^m \eta_{1l}^1 \tilde{\gamma}_k^{l+n} & \cdots & \sum_{l=1}^m \eta_{ml}^1 \tilde{\gamma}_k^{l+n} \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^m \eta_{1l}^m \tilde{\gamma}_k^{l+n} & \cdots & \sum_{l=1}^m \eta_{ml}^m \tilde{\gamma}_k^{l+n} \end{pmatrix} \in M_m(E) \quad (1 \leq k \leq n),$$

$$C_k^{(1)} := \begin{pmatrix} \sum_{l=1}^n \lambda_{1l}^1 \tilde{\gamma}_{k+n}^l & \cdots & \sum_{l=1}^n \lambda_{nl}^1 \tilde{\gamma}_{k+n}^l \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^n \lambda_{1l}^n \tilde{\gamma}_{k+n}^l & \cdots & \sum_{l=1}^n \lambda_{nl}^n \tilde{\gamma}_{k+n}^l \end{pmatrix} \in M_n(E) \quad (1 \leq k \leq m)$$

and

$$C_k^{(2)} := \begin{pmatrix} \sum_{l=1}^m \eta_{1l}^1 \tilde{\gamma}_{k+n}^{l+n} & \cdots & \sum_{l=1}^m \eta_{ml}^1 \tilde{\gamma}_{k+n}^{l+n} \\ \vdots & \ddots & \vdots \\ \sum_{l=1}^m \eta_{1l}^m \tilde{\gamma}_{k+n}^{l+n} & \cdots & \sum_{l=1}^m \eta_{ml}^m \tilde{\gamma}_{k+n}^{l+n} \end{pmatrix} \in M_m(E) \quad (1 \leq k \leq m).$$

Suppose that ψ is a twisting map. The representation $\hat{\rho}_\psi: D^{\text{op}} \longrightarrow M_{m+n}(E)$ of Corollary 2.2 is given by

$$\hat{\rho}_\psi(b_k^{\text{op}}) = \begin{pmatrix} B_k^{(1)} & 0 \\ 0 & B_k^{(2)} \end{pmatrix} \quad (1 \leq k \leq n) \quad (3.3)$$

and

$$\hat{\rho}_\psi(c_k^{\text{op}}) = \begin{pmatrix} C_k^{(1)} & 0 \\ 0 & C_k^{(2)} \end{pmatrix} \quad (1 \leq k \leq m). \quad (3.4)$$

Lemma 3.1. *The map $\hat{\rho}_\psi: D^{\text{op}} \longrightarrow M_{m+n}(E)$ defined by (3.3)-(3.4) is a representation if and only if the following conditions are satisfied:*

$$B_i^{(l)} B_j^{(l)} = \sum_{k=1}^n \lambda_{ji}^k B_k^{(l)} \quad \text{for } l \in \{1, 2\} \text{ and } 1 \leq i, j \leq n, \quad (3.5)$$

$$C_i^{(l)} C_j^{(l)} = \sum_{k=1}^m \eta_{ji}^k C_k^{(l)} \quad \text{for } l \in \{1, 2\} \text{ and } 1 \leq i, j \leq m, \quad (3.6)$$

$$B_i^{(l)} C_j^{(l)} = C_j^{(l)} B_i^{(l)} = 0 \quad \text{for } l \in \{1, 2\}, 1 \leq i \leq n \text{ and } 1 \leq j \leq m, \quad (3.7)$$

$$\sum_{i=1}^n \alpha_i B_i^{(1)} + \sum_{j=1}^m \beta_j C_j^{(1)} = \text{id}_{M_n(E)} \quad (3.8)$$

and

$$\sum_{i=1}^n \alpha_i B_i^{(2)} + \sum_{j=1}^m \beta_j C_j^{(2)} = \text{id}_{M_m(E)}. \quad (3.9)$$

Proof. This follows from a direct computation. \square

The representation $\hat{\phi}_\psi: A \longrightarrow M_{m+n}(A)$ of Proposition 2.5 is given by

$$\hat{\phi}_\psi(a) = \begin{pmatrix} \tilde{\gamma}_1^1(a) & \cdots & \tilde{\gamma}_{n+m}^1(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_1^{n+m}(a) & \cdots & \tilde{\gamma}_{n+m}^{n+m}(a) \end{pmatrix} \quad \text{for } a \in A, \quad (3.10)$$

Write

$$\hat{\phi}_\chi(a) = \begin{pmatrix} \Gamma_0^0(a) & \Gamma_1^0(a) \\ \Gamma_0^1(a) & \Gamma_1^1(a) \end{pmatrix},$$

with $\Gamma_0^0(a) \in M_n(K)$, $\Gamma_1^0(a) \in M_{n \times m}(K)$, $\Gamma_0^1(a) \in M_{m \times n}(K)$ and $\Gamma_1^1(a) \in M_m(K)$.

Remark 3.2. Equality (3.2) shows that $\Gamma_1^0 = 0$ if and only if $\psi(A \otimes C) \subseteq C \otimes A$.

Lemma 3.3. *The map $\hat{\phi}_\psi: A \longrightarrow M_{m+n}(A)$ defined by 3.10, is a representation if and only if the following conditions are satisfied:*

$$\Gamma_0^0(1) = I_n, \quad \Gamma_1^1(1) = I_m, \quad \Gamma_1^0(1) = 0, \quad \Gamma_0^1(1) = 0 \quad (3.11)$$

and

$$\Gamma_q^p(aa') = \Gamma_0^p(a) \Gamma_q^0(a') + \Gamma_1^p(a) \Gamma_q^1(a') \quad \text{for } a, a' \in A \text{ and } 0 \leq p, q \leq 1. \quad (3.12)$$

Proof. Straightforward. \square

Lemma 3.4. *Let $\psi: A \otimes D \longrightarrow D \otimes A$ be a map and let $\tilde{\gamma}_i^j$ ($1 \leq i, j \leq m+n$) be the maps determined by the equality (3.2). Let $i_B: B \rightarrow D$ and $p_B: D \rightarrow B$ be the canonical inclusion and the canonical surjection, respectively, and suppose that the map $\Theta := (p_B \otimes A) \circ \psi \circ (A \otimes i_B)$ is a twisting map. Then ψ is a twisting map if and only if the following conditions are satisfied*

$$B_i^{(2)} B_j^{(2)} = \sum_{k=1}^n \lambda_{ji}^k B_k^{(2)} \quad \text{for } 1 \leq i, j \leq n, \quad (3.13)$$

$$C_j^{(1)} = 0, \quad (3.14)$$

$$C_i^{(2)} C_j^{(2)} = \sum_{k=1}^m \eta_{ji}^k C_k^{(2)} \quad \text{for } 1 \leq i, j \leq m, \quad (3.15)$$

$$B_i^{(2)} C_j^{(2)} = C_j^{(2)} B_i^{(2)} = 0 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m, \quad (3.16)$$

$$\sum_{i=1}^n \alpha_i B_i^{(2)} + \sum_{j=1}^m \beta_j C_j^{(2)} = \text{id}_{M_m(E)}, \quad (3.17)$$

$$\Gamma_1^p(aa') = \Gamma_0^p(a) \Gamma_1^0(a') + \Gamma_1^p(a) \Gamma_1^1(a') \quad \text{for } a, a' \in A \text{ and } 0 \leq p \leq 1, \quad (3.18)$$

$$\Gamma_0^1(aa') = \Gamma_0^1(a) \Gamma_0^0(a') + \Gamma_1^1(a) \Gamma_0^1(a') \quad \text{for } a, a' \in A, \quad (3.19)$$

$$\Gamma_1^0(a) \Gamma_0^1(a') = 0 \quad \text{for } a, a' \in A \quad (3.20)$$

and

$$\Gamma_1^1(1) = I_m, \quad \Gamma_1^0(1) = 0, \quad \Gamma_0^1(1) = 0. \quad (3.21)$$

Proof. By Corollary 2.2, Proposition 2.5, Lemma 3.1 and Lemma 3.3, we know that ψ is a twisting map if and only if conditions (3.5)-(3.9), (3.11) and (3.12) are fulfilled. Since $\Gamma_0^0(a) = \hat{\phi}_\Theta(a)$ and Θ is a twisting map, we have

$$\Gamma_0^0(aa') = \Gamma_0^0(a)\Gamma_0^0(a') \quad \text{and} \quad \Gamma_0^0(1) = I_n.$$

Hence conditions (3.11) and (3.12) are equivalent to conditions (3.18)-(3.21). Moreover, again since Θ is a twisting map and $B_i^{(1)} = \hat{\rho}_\Theta(b_i^{op})$, we have

$$B_i^{(1)}B_j^{(1)} = \sum_{k=1}^n \lambda_{ji}^k B_k^{(1)} \quad \text{and} \quad \sum_{i=1}^n \alpha_i B_i^{(1)} = \text{id}_{M_n(E)}.$$

So, condition (3.5) is satisfied for $l = 1$, and condition (3.8) becomes equivalent to

$$\sum_{j=1}^m \beta_j C_j^{(1)} = 0.$$

Consequently, if ψ is a twisting map, then

$$\hat{\rho}_\psi(1_C) = \sum_{j=1}^m \beta_j \hat{\rho}_\psi(c_j) = \sum_{j=1}^m \beta_j \begin{pmatrix} 0 & 0 \\ 0 & C_j^{(2)} \end{pmatrix},$$

since $\hat{\rho}_\psi$ is a representation, and so $C_j^{(1)} = 0$. Conversely, if conditions (3.13)-(3.21) are fulfilled and Θ is a twisting map, then a direct computation shows that conditions (3.5)-(3.9), (3.11) and (3.12) are satisfied, which proves that ψ is a twisting map \square

Proposition 3.5. *Let $\psi: A \otimes D \longrightarrow D \otimes A$ be a map and let $\tilde{\gamma}_i^j$ ($1 \leq i, j \leq m+n$) be the maps determined by the equality (3.2). Let $i_B: B \rightarrow D$ and $p_B: D \rightarrow B$ be the canonical inclusion and the canonical surjection, respectively, and suppose that the map $\Theta := (p_B \otimes A) \circ \psi \circ (A \otimes i_B)$ is a twisting map. Then ψ is a twisting map if and only if the following conditions are satisfied*

$$B_i^{(2)}B_j^{(2)} = \sum_{k=1}^n \lambda_{ji}^k B_k^{(2)} \quad \text{for } 1 \leq i, j \leq n, \quad (3.22)$$

$$C_j^{(1)} = 0, \quad (3.23)$$

$$C_i^{(2)}C_j^{(2)} = \sum_{k=1}^m \eta_{ji}^k C_k^{(2)} \quad \text{for } 1 \leq i, j \leq m, \quad (3.24)$$

$$B_i^{(2)}C_j^{(2)} = C_j^{(2)}B_i^{(2)} = 0 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m, \quad (3.25)$$

$$\sum_{i=1}^n \alpha_i B_i^{(2)} + \sum_{j=1}^m \beta_j C_j^{(2)} = \text{id}_{M_m(E)}, \quad (3.26)$$

$$\Gamma_1^0 = 0, \quad (3.27)$$

$$\Gamma_1^1(aa') = \Gamma_1^1(a)\Gamma_1^1(a') \quad \text{for } a, a' \in A, \quad (3.28)$$

$$\Gamma_0^1(aa') = \Gamma_0^1(a)\Gamma_0^0(a') + \Gamma_1^1(a)\Gamma_0^1(a') \quad \text{for } a, a' \in A, \quad (3.29)$$

and

$$\Gamma_1^1(1) = I_m, \quad \Gamma_0^1(1) = 0. \quad (3.30)$$

Proof. \Leftarrow) By Lemma 3.4, this is clear.

\Rightarrow) Again by Lemma 3.4 it suffices to check that $\Gamma_1^0 = 0$ or, equivalently, that $\psi(A \otimes C) \subseteq C \otimes A$. Module a basis change we can assume that $b_1 = 1_B$. In this case, since $C_i^{(1)}$ we have:

$$\tilde{\gamma}_{i+n}^j = \sum_{l=1}^n \delta_{jl} \tilde{\gamma}_{i+n}^l = \sum_{l=1}^n \lambda_{1l}^j \tilde{\gamma}_{i+n}^l = (C_i^{(1)})_{j1} = 0$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, as desired. \square

Remark 3.6. For each $a \in A$, write

$$\begin{aligned} \varphi_B(a) &:= \begin{pmatrix} \tilde{\gamma}_1^1(a) & \dots & \tilde{\gamma}_n^1(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_1^n(a) & \dots & \tilde{\gamma}_n^n(a) \end{pmatrix}, \\ \varphi_C(a) &:= \begin{pmatrix} \tilde{\gamma}_{n+1}^{n+1}(a) & \dots & \tilde{\gamma}_{n+m}^{n+1}(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n+1}^{n+m}(a) & \dots & \tilde{\gamma}_{n+m}^{n+m}(a) \end{pmatrix} \end{aligned}$$

and

$$\Delta(a) := \begin{pmatrix} \tilde{\gamma}_1^{n+1}(a) & \dots & \tilde{\gamma}_n^{n+1}(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_1^{n+m}(a) & \dots & \tilde{\gamma}_n^{n+m}(a) \end{pmatrix}.$$

If the hypothesis of Proposition 3.5 are satisfied, then

$$\varphi_\psi(a) = \begin{pmatrix} \varphi_B(a) & 0 \\ \Delta(a) & \varphi_C(a) \end{pmatrix} \quad \text{for all } a \in A. \quad (3.31)$$

Furthermore, since φ_ψ is a representation, the following equalities hold

$$\begin{aligned} \varphi_B(aa') &= \varphi_B(a)\varphi_B(a'), \\ \varphi_C(aa') &= \varphi_C(a)\varphi_C(a') \end{aligned}$$

and

$$\Delta(aa') = \Delta(a)\varphi_B(a') + \varphi_C(a)\Delta(a'),$$

for all $a, a' \in A$.

Corollary 3.7. *Suppose that the hypothesis of Proposition 3.4 are satisfied. Let $i_C: C \rightarrow D$ and $p_C: D \rightarrow C$ be the canonical inclusion and the canonical surjection, respectively. If the map $\Upsilon := (p_C \otimes A) \circ \psi \circ (A \otimes i_C)$ is a twisting map. Then ψ is a twisting map if and only if $\psi = \Theta \oplus \Upsilon$*

Proof. Use Proposition 3.5 and Remark 3.2 applied to B and C . \square

4 Change of Basis

The representations introduced in Section 2 depend on the choice of the basis in B . In this section we analyze how the representations φ_χ and $\hat{\rho}_\chi$ behave under a base change.

Let A, B and C algebras over a field K and let $f: B \rightarrow C$ be a morphism of algebras. Assume that B and C are finite dimensional and fix basis $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_m\}$ of B and C , respectively. Consider the matrix

$$M_{\mathcal{B}}^{\mathcal{C}}(f) := \begin{pmatrix} \zeta_1^1 \cdot 1_A & \dots & \zeta_n^1 \cdot 1_A \\ \vdots & \ddots & \vdots \\ \zeta_1^m \cdot 1_A & \dots & \zeta_n^m \cdot 1_A \end{pmatrix} \in M_{m \times n}(A),$$

where the scalars $\zeta_i^j \in K$ are determined by the equalities

$$f(b_j) = \sum_{i=1}^m \zeta_j^i \cdot c_i.$$

Proposition 4.1. *Let $\chi: A \otimes B \rightarrow B \otimes A$ and $\varpi: A \otimes C \rightarrow C \otimes A$ be twisting maps. The map*

$$f \otimes A: B \otimes_\chi A \rightarrow C \otimes_\varpi A$$

is a morphism of algebras if and only if

$$\varphi_\varpi(a) M_{\mathcal{B}}^{\mathcal{C}}(f) = M_{\mathcal{B}}^{\mathcal{C}}(f) \varphi_\chi(a) \quad \text{for all } a \in A, \quad (4.32)$$

where φ_ϖ and φ_χ are the representations defined in Corollary 2.9.

Proof. Let $\gamma_i^j: A \rightarrow A$ ($1 \leq i, j \leq n$) and $\tilde{\gamma}_i^j: A \rightarrow A$ ($1 \leq i, j \leq m$) be the maps determined by the equality (1.1) applied to χ and ϖ , respectively. A direct computation using Remark 1.2 shows that $f \otimes A$ is a morphism if and only if

$$\sum_{i=1}^n \zeta_i^j \cdot \gamma_k^i(a) = \sum_{u=1}^m \zeta_k^u \cdot \tilde{\gamma}_u^j(a) \quad \text{for all } a \in A. \quad (4.33)$$

But this happens if and only if condition (4.32) is fulfilled. \square

Remark 4.2. Let

$$\widehat{M}_{\mathcal{B}}^{\mathcal{C}}(f) := \begin{pmatrix} \zeta_1^1 \cdot \text{id} & \dots & \zeta_n^1 \cdot \text{id} \\ \vdots & \ddots & \vdots \\ \zeta_1^m \cdot \text{id} & \dots & \zeta_n^m \cdot \text{id} \end{pmatrix} \in M_{m \times n}(E).$$

A direct computation using (4.32) shows that under the hypothesis of Proposition 4.1, we have

$$\hat{\rho}_\varpi(f(b)^{op}) \widehat{M}_{\mathcal{B}}^{\mathcal{C}}(f) = \widehat{M}_{\mathcal{B}}^{\mathcal{C}}(f) \hat{\rho}_\chi(b^{op}),$$

for all $b \in B$.

Corollary 4.3. *Let $\mathcal{B}' := \{b'_1, \dots, b'_n\}$ be another basis of B and let φ'_χ and $\hat{\rho}'_\chi$ be the representations associated to χ according Corollaries 2.2 and 2.9, but using the basis \mathcal{B}' instead of \mathcal{B} . Let $M := M_{\mathcal{B}}^{\mathcal{B}'}(\text{id}_B)$ and $\widehat{M} := \widehat{M}_{\mathcal{B}}^{\mathcal{B}'}(\text{id}_B)$. Then*

$$\varphi'_\chi(a) = M \varphi_\chi(a) M^{-1} \quad \text{and} \quad \hat{\rho}'_\chi(b^{op}) = \widehat{M} \hat{\rho}_\chi(b^{op}) \widehat{M}^{-1}, \text{ for all } a \in A \text{ and } b \in B.$$

Proof. A straightforward computation using Proposition 4.32 and Remark 4.2. \square

5 Examples

5.1 Non-commutative duplicates of finite sets

Consider the algebra $B = \frac{K[X]}{\langle X^2 - X \rangle}$ with the basis $\mathcal{B} = \{1, X\}$. The structure matrices of 1 and X with respect to (B, \mathcal{B}) (see Definition 2.3) are the matrices

$$[X]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad [1]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider a map $\chi: A \otimes B \rightarrow B \otimes A$. Proceeding as in Subsection 1.2 we determine maps $\gamma_2^1: A \rightarrow A$ and $\gamma_2^2: A \rightarrow A$ such that

$$\chi(a \otimes X) = 1 \otimes \gamma_2^1(a) + X \otimes \gamma_2^2(a).$$

By Corollary 2.2 and Proposition 2.5 we know that χ is a twisting map if and only if the maps $\hat{\rho}_\chi: B^{op} \rightarrow M_n(E)$ and $\hat{\phi}_\chi: A \rightarrow M_n(A)$ are matrix representations. The associated twisted tensor products $A \otimes_\chi B$ were studied in [3], where they are called *Non-Commutative Duplicates*

(Actually they studied twisted tensor products $B \otimes_\chi A$, which gives the same results, taking the opposite algebras).

The map $\hat{\rho}_B: B^{\text{op}} \rightarrow M_2(E)$ is given by the matrices

$$\hat{\rho}_B(X) = \begin{pmatrix} \gamma_2^1 & 0 \\ \gamma_2^2 & \gamma_2^1 + \gamma_2^2 \end{pmatrix} \quad \text{and} \quad \hat{\rho}_B(1) = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}.$$

The equalities $\hat{\rho}_B(X) = \hat{\rho}_B(X^2) = \hat{\rho}_B(X)^2$ determine the conditions

- (1) $\gamma_2^1 \circ \gamma_2^1 = \gamma_2^1$,
- (2) $\gamma_2^2 \circ \gamma_2^1 + \gamma_2^1 \circ \gamma_2^2 + \gamma_2^2 \circ \gamma_2^2 = \gamma_2^2$,
- (3) $(\gamma_2^1 + \gamma_2^2)^2 = \gamma_2^1 + \gamma_2^2$,

that the maps γ_2^1, γ_2^2 must satisfy in order that $\hat{\rho}_B$ be a representation.

On the other hand the map $\hat{\phi}_\chi$ is given by the matrices

$$\hat{\phi}_\chi(a) := \begin{pmatrix} a & \gamma_2^1(a) \\ 0 & \gamma_2^2(a) \end{pmatrix} \quad \text{for } a \in A.$$

A direct computation shows that $\hat{\phi}_\chi$ is a representation if and only if the maps γ_2^1 and γ_2^2 satisfy

- (4) $\gamma_2^1(1) = 0$,
- (5) $\gamma_2^2(1) = 1$,
- (6) $\gamma_2^1(ab) = a\gamma_2^1(b) + \gamma_2^1(a)\gamma_2^2(b)$,
- (7) $\gamma_2^2(ab) = \gamma_2^2(a)\gamma_2^2(b)$.

It is easy to see that (1) and (2) imply (3), while (5) and (6) imply (4). Therefore, the above conditions are satisfied if and only if $f := \gamma_2^2$ is an endomorphism of A and $\delta := \gamma_2^1$ is an (id, f) -derivation satisfying $f = f^2 + \delta \circ f + f \circ \delta$ (Compare with [3, Definition 2.7]).

Finally, the representation φ_χ in Corollary 2.9 is given by the matrices

$$\varphi_\chi(a) := \begin{pmatrix} a & \gamma_2^1(a) \\ 0 & \gamma_2^2(a) \end{pmatrix} \quad \text{for } a \in A, \quad \varphi_\chi(X) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \varphi_\chi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.2 Factorizations structures with a two-dimensional factor

Given a polynomial $P(X) := X^2 - \alpha X + \beta \in K[X]$ consider the algebra $B = \frac{K[X]}{\langle P(X) \rangle}$ with the basis $\mathcal{B} = \{1, X\}$. The structure matrices of 1 and X with respect to (B, \mathcal{B}) are the matrices

$$[X]_{\mathcal{B}} = \begin{pmatrix} 0 & -\beta \\ 1 & \alpha \end{pmatrix} \quad \text{and} \quad [1]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider a map $\chi: A \otimes B \rightarrow B \otimes A$. Proceeding as above we determine maps $\gamma_2^1: A \rightarrow A$ and $\gamma_2^2: A \rightarrow A$ such that

$$\chi(a \otimes X) = 1 \otimes \gamma_2^1(a) + X \otimes \gamma_2^2(a).$$

As above, χ is a twisting map if and only if the maps $\hat{\rho}_\chi: B^{\text{op}} \rightarrow M_n(E)$ and $\hat{\phi}_\chi: A \rightarrow M_n(A)$ are matrix representations. The associated twisted tensor products $A \otimes_\chi B$ (respectively $B \otimes_\chi A$) were studied in [8], where they are called *quantum duplicates*. The maps f and δ considered in that paper correspond to the maps γ_2^2 and γ_2^1 , respectively.

The map $\hat{\rho}_B: B^{\text{op}} \rightarrow M_2(E)$ is given by the matrices

$$\hat{\rho}_B(X) = \begin{pmatrix} \gamma_2^1 & -\beta\gamma_2^2 \\ \gamma_2^2 & \gamma_2^1 + \alpha\gamma_2^2 \end{pmatrix} \quad \text{and} \quad \hat{\rho}_B(1) = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}.$$

The equalities $\hat{\rho}_B(X)^2 = \hat{\rho}_B(X^2) = \alpha\hat{\rho}_B(X) - \beta\hat{\rho}_B(1)$ determine the conditions

- (1) $\gamma_2^1 \circ \gamma_2^1 - \beta \gamma_2^2 \circ \gamma_2^2 = \alpha \gamma_2^1 - \beta \text{id},$
- (2) $\gamma_2^1 \circ \gamma_2^2 + \gamma_2^2 \circ \gamma_2^1 + \alpha \gamma_2^2 \circ \gamma_2^2 = \alpha \gamma_2^2,$

that the maps γ_2^1, γ_2^2 must satisfy in order that $\hat{\rho}_B$ is a representation.

On the other hand the map $\hat{\phi}_\chi$ is given by the matrices

$$\hat{\phi}_\chi(a) = \begin{pmatrix} a & \gamma_2^1(a) \\ 0 & \gamma_2^2(a) \end{pmatrix} \quad \text{for } a \in A.$$

A direct computation shows that $\hat{\phi}_\chi$ is a representation if and only if the maps γ_2^1 and γ_2^2 satisfy

- (3) $\gamma_2^2(1) = 1,$
- (4) $\gamma_2^1(ab) = a\gamma_2^1(b) + \gamma_2^1(a)\gamma_2^2(b),$
- (5) $\gamma_2^2(ab) = \gamma_2^2(a)\gamma_2^2(b).$

Therefore, the above conditions are satisfied if and only if $f := \gamma_2^2$ is an endomorphism of A and $\delta := \gamma_2^1$ is an (id, f) -derivation satisfying

$$P(\delta) = \beta f^2 \quad \text{and} \quad f \circ \delta + \delta \circ f = \alpha(f - f^2)$$

(Compare with [8, Lemma 1.1]).

Finally the representation φ_χ in Corollary 2.9 is given by the matrices

$$\varphi_\chi(a) = \begin{pmatrix} a & \gamma_2^1(a) \\ 0 & \gamma_2^2(a) \end{pmatrix} \quad \text{for } a \in A, \quad \varphi_\chi(X) = \begin{pmatrix} 0 & -\beta \\ 1 & \alpha \end{pmatrix} \quad \text{and} \quad \varphi_\chi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.3 Twisting with K^n

Let $B := K^n$ and let $\mathcal{B}_2 = \{e_1, \dots, e_n\}$ be the canonical basis of B . The structure matrices of e_1, \dots, e_n with respect to (B, \mathcal{B}) are the matrices e_{11}, \dots, e_{nn} , where e_{ii} is the matrix with 1 in the i -th entry of the main diagonal and 0 in the other entries. Given a twisted tensor product $\chi: B \otimes A \longrightarrow A \otimes B$ we have maps $\tilde{\gamma}_i^j: A \longrightarrow A$ ($1 \leq i, j \leq n$) defined by the equalities

$$\chi(a \otimes e_i) = \sum_{j=1}^n e_1 \otimes \tilde{\gamma}_i^j(a).$$

As above, χ is a twisting map if and only if the maps $\hat{\rho}_\chi: B^{\text{op}} \longrightarrow M_n(E)$ and $\hat{\phi}_\chi: A \longrightarrow M_n(A)$ are matrix representations. The associated twisted tensor products $A \otimes_\chi B$ were studied in [7]. The maps E_{ji} considered in that paper correspond to the maps $\tilde{\gamma}_j^i$.

The map $\hat{\rho}_B: B^{\text{op}} \longrightarrow M_n(E)$ is given by the matrices

$$\hat{\rho}_B(e_i) = \begin{pmatrix} \tilde{\gamma}_i^1 & 0 & \dots & 0 \\ 0 & \tilde{\gamma}_i^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\gamma}_i^n \end{pmatrix}.$$

The equalities

$$\hat{\rho}_B(e_i)^2 = \hat{\rho}_B(e_i), \quad \hat{\rho}_B(e_i)\hat{\rho}_B(e_j) = 0 \quad \text{and} \quad \hat{\rho}_B(e_1) + \dots + \hat{\rho}_B(e_n) = \text{id}$$

determine the conditions

- (1) $\tilde{\gamma}_i^p \circ \tilde{\gamma}_j^p = \delta_{ij} \tilde{\gamma}_i^p$ ($1 \leq i, j, p \leq n$),
- (2) $\sum_{j=1}^n \tilde{\gamma}_j^i = \text{id}_A$ for $i = 1, \dots, n$,

that the maps $\tilde{\gamma}_{ij}^p$ must satisfy in order that $\hat{\rho}_B$ is a representation.

On the other hand the map $\hat{\phi}_\chi$ is given by the matrices

$$\hat{\phi}_\chi(a) = \begin{pmatrix} \tilde{\gamma}_1^1(a) & \dots & \tilde{\gamma}_n^1(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_n^1(a) & \dots & \tilde{\gamma}_n^n(a) \end{pmatrix} \quad \text{for } a \in A.$$

A direct computation shows that $\hat{\phi}_\chi$ is a representation if and only if the maps $\tilde{\gamma}_j^i$ satisfy

- (3) $\tilde{\gamma}_j^i(ab) = \sum_{p=1}^n \tilde{\gamma}_p^i(a) \tilde{\gamma}_j^p(b)$ for $i, j = 1, \dots, n$,
- (4) $\tilde{\gamma}_j^i(1_A) = \delta_{ij} 1_A$ for $i, j = 1, \dots, n$.

Conditions (1)–(4) correspond to the conditions (6)–(9) in [7].

When $n = 2$, we have the isomorphism $\frac{k[X]}{(X^2 - X)} \simeq K^2$ which sends X to e_2 . Using Proposition 4.1 is straightforward to check that the maps f and δ in the first example satisfy

$$f = \tilde{\gamma}_2^2 - \tilde{\gamma}_2^1 \quad \text{and} \quad \delta = \tilde{\gamma}_2^1.$$

The matrix representation $\hat{\phi}_\chi$ can be used to define a pair $(Q_\chi, \mathcal{R}_\chi)$ where Q_χ is a quiver and \mathcal{R}_χ is a representation of the quiver Q_χ , as follows

Definition 5.1. The Quiver Q_χ is defined as follows: the set of vertices of Γ_χ is $\{v_1, \dots, v_n\}$. The vertices v_i and v_j are joined by an arrow with the source v_j if and only if $\gamma_j^i \neq 0$

Definition 5.2. Let Q_χ be the quiver associated to $\hat{\phi}_\chi$. The representation \mathcal{R}_χ of Q_χ is defined by the family of vector spaces $\{V_i\}_{i \in Q_\chi^0}$ and the family of K -linear maps $\{f_\alpha\}_{\alpha \in Q_\chi^1}$, where

$$V_i = A, \quad f_\alpha = \gamma_j^i: V_j \rightarrow V_i, \quad j = s(\alpha), i = t(\alpha)$$

Remark 5.3. The map χ is a twisting map if and only if the pair $(Q_\chi, \mathcal{R}_\chi)$ es admissible of order n (see [7, Definition 1.7])

- (1) The conditions (1)–(2) correspond to splitted condition (see [7, Proposition 1.6])
- (2) The condition (3) corresponds to unital condition (see [7, Proposition 1.6])
- (3) The condition (4) corresponds to factorizable condition (see [7, Proposition 1.6])

5.4 Extensions of K^m to K^n

Consider a twisting map $\chi: A \otimes K^n \longrightarrow K^n \otimes A$ defined by $\chi(a \otimes e_j) = \sum_{i=1}^n e_i \otimes \tilde{\gamma}_j^i(a)$ and suppose that exists $m < n$ such that the map $\Theta: A \otimes K^m \longrightarrow K^m \otimes A$ defined by $\Theta(a \otimes e_j) = \sum_{i=1}^m e_i \otimes \tilde{\gamma}_j^i(a)$ for $j \in \{1, \dots, m\}$ is a twisting map. Write

$$C_j^{(1)} = \hat{\rho}_B(e_j) = \begin{pmatrix} \tilde{\gamma}_j^1 & 0 & \dots & 0 \\ 0 & \tilde{\gamma}_j^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\gamma}_j^m \end{pmatrix}, \quad \text{for } j \in \{m+1, \dots, n\}.$$

Then, by proposition 3.4, we have $C_j^{(1)} = 0$ for $j \in \{m+1, \dots, n\}$, consequently $\tilde{\gamma}_j^i = 0$ for $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$. Therefore, the representation $\hat{\phi}_\chi$ looks as follows

$$\hat{\phi}_\chi(a) = \begin{pmatrix} \hat{\phi}_\Theta(a) & 0 \\ \Delta(a) & \hat{\phi}_{K^{n-m}}(a) \end{pmatrix} \quad \text{para todo } a \in A, \quad (5.34)$$

where

$$\Delta(a) = \begin{pmatrix} \tilde{\gamma}_1^{m+1}(a) & \cdots & \tilde{\gamma}_m^{m+1}(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_1^n(a) & \cdots & \tilde{\gamma}_m^n(a) \end{pmatrix} \quad \text{and} \quad \varphi_{K^{n-m}}(a) = \begin{pmatrix} \tilde{\gamma}_{m+1}^{m+1}(a) & \cdots & \tilde{\gamma}_n^{m+1}(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_{m+1}^n(a) & \cdots & \tilde{\gamma}_n^n(a) \end{pmatrix},$$

for $a \in A$.

In particular, the twisted tensor products that come from a Quiver of rank 1 with a cycle of length 2 studied in [7, Theorem 4.6] are extensions of a twisted tensor product generated by the twisting map $\Theta: A \otimes K^2 \rightarrow K^2 \otimes A$.

On the other hand, when the map $\Upsilon: A \otimes K^{n-m} \rightarrow K^{n-m} \otimes A$ defined by $\Upsilon(a \otimes e_j) = \sum_{i=m+1}^n e_i \otimes \tilde{\gamma}_j^i(a)$ for $j \in \{m+1, \dots, n\}$ is a twisting map. Write

$$B_j^{(2)} = \hat{\rho}_B(e_j) = \begin{pmatrix} \tilde{\gamma}_j^{m+1} & 0 & \cdots & 0 \\ 0 & \tilde{\gamma}_j^{m+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\gamma}_j^n \end{pmatrix}, \quad \text{for } j \in \{1, \dots, m\}.$$

Then, by corollary 3.7, we have $B_j^{(2)} = 0$ for $j \in \{1, \dots, m\}$, consequently $\tilde{\gamma}_j^i = 0$ for $i \in \{m+1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Therefore, the representation φ_χ looks as follows

$$\varphi_\chi(a) = \begin{pmatrix} \varphi_\Theta(a) & 0 \\ 0 & \varphi_\Upsilon(a) \end{pmatrix} \quad \text{para todo } a \in A, \quad (5.35)$$

The twisted tensor products for which $m = 2$ and Υ come from a Quiver of rank 1 without cycles of length 2 studied in [7, Theorem 4.6] are extensions of a twisted tensor product generated by the twisting map $\Theta: A \otimes K^2 \rightarrow K^2 \otimes A$. By example, the twisting map for which $\varphi_\Upsilon = \text{id}$ where studied in [7, Corollary 4.3].

Now we construct extensions from K^2 to K^3 . Consider a twisting map $\chi: K^3 \otimes K^3 \rightarrow K^3 \otimes K^3$ defined by

$$\chi(a \otimes e_j) = e_1 \otimes \tilde{\gamma}_j^1(a) + e_2 \otimes \tilde{\gamma}_j^2(a) + e_3 \otimes \tilde{\gamma}_j^3(a) \quad j \in \{1, 2, 3\}$$

and suppose that the map $\Theta: K^3 \otimes K^2 \rightarrow K^2 \otimes K^3$ defined by

$$\Theta(a \otimes e_j) = e_1 \otimes \tilde{\gamma}_j^1(a) + e_2 \otimes \tilde{\gamma}_j^2(a) \quad j \in \{1, 2\}$$

is a twisting map. Then, by Proposition 3.5, we know that $\tilde{\gamma}_3^1 = \tilde{\gamma}_3^2 = 0$.

5.5 Noncommutative truncated polynomial extensions

Let $B := \frac{K[Y]}{\langle Y^n \rangle}$ with the basis $\mathcal{B} = \{1, Y, Y^2, \dots, Y^{n-1}\}$. The structure matrices of 1 and Y with respect to (B, \mathcal{B}) are the matrices

$$[Y]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad [1]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Given a map $\chi: B \otimes A \rightarrow A \otimes B$ we have unique maps $\tilde{\gamma}_i^j: A \rightarrow A$ ($0 \leq i, j \leq n-1$) such that

$$\chi(a \otimes Y^i) = \sum_{j=0}^{n-1} Y^j \otimes \tilde{\gamma}_i^j(a).$$

We know that χ is a twisting map if and only if $\hat{\rho}_\chi: B^{\text{op}} \rightarrow M_n(E)$ and $\hat{\phi}_\chi: A \rightarrow M_n(A)$ are matrix representations. The associated twisted tensor products $A \otimes_\chi B$ were studied in [5].

The maps $\gamma_i^j: A^{\text{op}} \longrightarrow A^{\text{op}}$ considered in that paper are related to the maps $\tilde{\gamma}_j^i: A \rightarrow A$ by the equalities $\gamma_i^j(a^{\text{op}}) = \tilde{\gamma}_j^i(a)$ for $0 \leq i, j < n$ and all $a \in A$.

The map $\hat{\rho}_B: B^{\text{op}} \longrightarrow M_n(E)$ is given by the matrices

$$\hat{\rho}_B(Y^i) = \begin{pmatrix} \tilde{\gamma}_i^0 & 0 & 0 & \cdots & 0 \\ \tilde{\gamma}_i^1 & \tilde{\gamma}_i^0 & 0 & \cdots & 0 \\ \tilde{\gamma}_i^2 & \tilde{\gamma}_i^1 & \tilde{\gamma}_i^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_i^{n-1} & \tilde{\gamma}_i^{n-2} & \tilde{\gamma}_i^{n-3} & \cdots & \tilde{\gamma}_i^0 \end{pmatrix}.$$

The equalities

$$\hat{\rho}_B(Y^r) = \hat{\rho}_B(Y^{r-i})\hat{\rho}_B(Y^i), \quad \hat{\rho}_B(Y^n) = 0 \quad \text{and} \quad \hat{\rho}_B(1) = \text{id}$$

determine the conditions

- (1) $\tilde{\gamma}_0^j = \delta_{0j} \text{id}$,
- (2) $\tilde{\gamma}_r^j = \sum_{l=0}^j \tilde{\gamma}_{r-i}^{j-l} \circ \tilde{\gamma}_i^l$ for $j < n$, $1 < r < n$ and $0 < i < r$,
- (3) $\sum_{l=0}^j \tilde{\gamma}_{n-i}^{j-l} \circ \tilde{\gamma}_i^l = 0$ for $j < n$ and $0 < i < n$,

that the maps $\tilde{\gamma}_{ij}^p$ must satisfy in order that $\hat{\rho}_B$ is a representation.

On the other hand the map $\hat{\phi}_\chi$ is given by the matrices

$$\hat{\phi}_\chi(a) = \begin{pmatrix} \tilde{\gamma}_0^0(a) & \cdots & \tilde{\gamma}_{n-1}^0(a) \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_0^{n-1}(a) & \cdots & \tilde{\gamma}_{n-1}^{n-1}(a) \end{pmatrix} \quad \text{for } a \in A.$$

A direct computation shows that $\hat{\phi}_\chi$ is a representation if and only if the maps $\tilde{\gamma}_j^i$ satisfy

- (4) $\tilde{\gamma}_j^i(ab) = \sum_{p=1}^n \tilde{\gamma}_p^i(a)\tilde{\gamma}_j^p(b)$ for $i, j = 0, \dots, n-1$,
- (5) $\tilde{\gamma}_j^i(1_A) = \delta_{ij}1_A$ for $i, j = 0, \dots, n-1$.

Conditions (1)–(5) correspond to the conditions (2)(a)–(d) in [5, Proposition 1.2] and to the assumption $\gamma_j^r = 0$ for $r \geq n$ made in that paper.

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